Design Problem in Hydrodynamics

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The empirical procedures employed in designing hydrodynamic bodies are analyzed and interpreted as an iterative procedure. A solution is presented for one step of the iterative procedure and this is seen to be a solution to the development problem in which a small change in body shape is sought which will produce a specified small change in the pressure distribution. The development problem for a real incompressible fluid at high Reynolds' number is reduced to a free boundary problem in potential theory. The free boundary element is removed by a perturbation procedure and an analytical solution, using a surface source distribution, is obtained in the form of integral equations. These are solved numerically using techniques of the Douglas program. The results of a test case, in which the exact solution is known, are presented.

Nomenclature

g	=	body shape perturbation
l	=	field point-source point distance
p	=	pressure
r	=	radial coordinate
r_0	=	body radius
s	=	arc length
S	=	body surface
U	=	freestream velocity
v	=	tangential speed perturbation
V_T	=	tangential speed
W	=	pseudo-tangential speed
x	=	axial coordinate
X_{IJ}, Y_{IJ}	=	numerical coefficients
α	=	body slope
β	=	pseudo-normal velocity
ϵ	=	perturbation parameter
ν_j	=	unit normal vector
ρ	=	density
σ	=	source strength density
τ_j	=	unit tangent vector
ϕ	=	perturbation potential
Φ	=	potential
Δ	=	Laplacian
$(\bar{\ })$	=	datum flowfield quantities
$(\ )_{,k}$	=	partial derivative
$(\ )_\infty$	=	freestream

Subscripts

i,j,k	=	1,2
I,J,K	=	$1,2,\ldots N$

1. Introduction

THE design and development of hydrodynamic bodies such as torpedos, submarines, sonar domes, etc. is usually an empirical procedure, employing trial and error techniques, and leaning heavily upon the designer's experience. A desired objective is to reduce such empirical methods to deterministic ones. The possibility of accomplishing such an objective is increased by the availability of the computer such as in aircraft design.[1]

Two approaches to this objective are large scale parameter variation or, as is pursued here, the extraction from empirical procedures of fundamental problems which can be solved algorithmically (i.e., on a computer). From the point of view of hydrodynamic theory alone, it is possible to define two problems which have been seen to be fundamental. These are, given a surface pressure distribution, what is the corresponding body shape (the design problem)[2] and, second, given a slight change in the surface pressure distribution of an existing body, what is the corresponding slight change in the body shape (the development problem). The latter will be seen to be a subproblem of the former.

The role of the pressure distribution may be illustrated in the context of the design problem by the implication of a design constraint of minimum drag of a sonar dome, or, in the context of the development problem, the removal of a separation bubble in the neighborhood of a transducer.

Solutions to the design problem, available in the literature, are of an admittedly crude nature due to the absence of mathematical techniques for free boundary value problems, a situation which manifests itself in the prime characteristic of the design problem: in analytical form it is not known whether or not there exists a unique solution, continuously dependent upon given data. In physical terms, it is not known whether or not a given pressure distribution uniquely determines a body shape.[2] (It is to be noted that partial success has been achieved in the two-dimensional case, the Lighthill and the Goldstein methods,[2] due primarily to the availability of complex variable and hodograph techniques but even these solutions are not rigorous, usually requiring ad hoc steps stemming from experience of the designer.)

The method, considered to be the best available today, is that due to Young and Owen.[2,4,5] This method confines itself to slender bodies (and was found at a later date to be describable within the context of slender body theory) and to certain subsets of pressure distributions. In addition, the technique is admittedly nonrigorous. Now aside from the vagaries of the two latter characteristics, the restriction to slender bodies excludes the treatment of torpedo-like shapes with flat faces and other bodies not admitted to the slender body category.

A method presented at a later date by McKnown and Hsu[6] has been seen to be essentially the Young and Owen method.

Another attempt at solution is that of Munzer and Reichardt,[7] which was concerned primarily with determining bodies with constant pressure distributions (the cavity problem). Again slender bodies and ad hoc techniques obtain.

The aim of this paper is two fold: to present a solution to the development problem and to place this solution in the context of the larger, more difficult problem of design. Thus, an approach to the solution of the design problem is proposed implicitly. To this end, the design problem is considered first, the development problem is shown to be a subproblem

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of the former, and then its solution is presented in this context.

In the analyses to be presented, the design problem is considered in the context of real fluid flow and is reduced to a free boundary value problem in potential theory. A method of solution is proposed which may be considered to be a programing of existing design procedures. This is an iteration process in which one step is the solution of the development problem. The development problem is solved by means of surface source distribution, thus allowing for arbitrary body shapes, and utilizing the numerical techniques similar to those of the Douglas program.[3] The accuracy of the solution for the development problem is illustrated in a test case involving flow over a spheroid in which the exact solution is known. Work on the iteration phase is in progress.

2. Design Problem

The design problem, as defined previously, is to find the shape of a body traveling through a fluid if the resulting pressure distribution is given. Thus, to obtain a relationship between the pressure distribution and the body shape, consider a three-dimensional body moving at constant speed through a real fluid (e.g., water) at sufficiently low speeds (e.g., as is the case of a torpedo or submarine) such that the fluid may be assumed incompressible. In this context, the Reynolds' number is large such that the solutions to the governing equations (the Navier-Stokes equations), which include the pressure distributions over the body, may be expanded in a series in powers of the inverse Reynolds' number. The first problem encountered is the inviscid incompressible flow over the body at given speed with the solution valid outside the boundary layer.

The second problem encountered, using results of the first problem, is the classical boundary-layer problem, the solution of which is valid inside the boundary layer close to the body. This procedure is repeated. The resulting flowfield is a reflection of Prandtl's original boundary-layer hypothesis.

Consider the first problem: this is for the flow of an inviscid, incompressible fluid. The resulting velocity field determines a pressure distribution by means of the Bernoulli equation. But the second problem requires this pressure distribution to determine the characteristics of the boundary layer and, furthermore, implies that this pressure distribution is sensibly the pressure distribution acting upon the body.

Thus, the first problem encountered in relating pressure distribution and body shape in the context of real fluid theory is the flow of an inviscid incompressible fluid over the body traveling with given speed. Analytically, this requires a solution to Laplace's equation with the speed of travel given at infinity (for axes fixed in the body) and zero normal velocity components on the body.

Assuming an axisymmetric body with meridian curve $r = r_0(x)$ (see Fig. 1). The boundary value problem is

$$
\begin{aligned}
&\text{DE } \Delta\Phi(x,r) = 0 \\
&\text{BC 1) } r = r_0(x):\quad \Phi_{,j}\nu_j = 0 \\
&\quad\;\; 2)\ x^2 + r^2 \to \infty:\quad \Phi \to Ux \qquad (1) \\
&V_T = \Phi_{,j}\tau_j \\
&V_T^2/2 + p/\rho = U^2/2 + p_\infty/\rho
\end{aligned}
$$

Fig. 1 Flowfield.

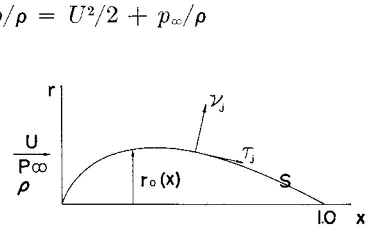
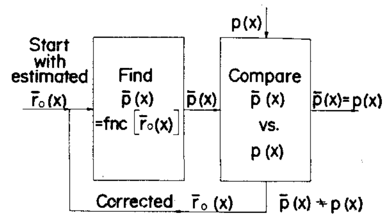


Fig. 2 Design procedure.



with $V_T(x)$ the surface tangential velocity and ν_j, τ_j unit normal and tangential vector at the body surface ($i,j = 1,2$).

For a given body shape, this is classified as a three dimensional, exterior, boundary value problem of the second kind in potential theory for which ample theory exists and for which excellent techniques for obtaining numerical solutions exists (the Douglas program[3]).

Now the first problem in the design procedure approached analytically is a variation of the previously well-posed problem in that the body shape $r_0(x)$ is not given but is to be found based upon a property of the solutions, namely a given surface pressure distribution, or via Bernoulli's equation, a given surface tangential velocity distribution $V_T(x)$. This is the design problem as defined previously and thus, it may be classified as a free boundary problem in potential theory for which little theory exists, particularly in three dimensions.

Obviously, separation of the boundary layer would nullify the solution to Eqs. (1) as a physically real solution. However, if matters are confined to flowfields in which separation is not catastrophic, then in view of the success of the Douglas program in describing actual flows over such bodies, the previous approach is a valid one.

3. Design Procedure

The preceding is an analytical formulation of the design problem. Obviously, design in this area does not rest upon available analytical method but employs various empirical procedures based primarily upon the individual designer's experience. The background of such procedures is now discussed and its program will be presented in the following section.

Thus, consider a proposed design involving an axisymmetric, three dimensional body with meridian curve $r_0(x)$, $0 \leq x \leq 1$, traveling at constant speed and experiencing a surface pressure distribution $p(x)$. In its essence, the designer is faced with the problem of finding the shape $r_0(x)$, given the surface pressure distribution $p(x)$ [or the surface velocity distribution $V_T(x)$]. The procedure he follows may be described by Fig. 2.

The designer starts with a guessed $r_0(x)$, say $\bar{r}_0(x)$, based upon his experience. He finds the corresponding $\bar{p}(x)$ [via $\bar{V}_T(x)$] either analytically (say, via the Douglas program[3]) or experimentally since this is a deterministic problem. He then compares the resulting $\bar{p}(x)$ with the given $p(x)$. Based upon his experience, he then accepts or corrects the initial guess and repeats the process until an acceptable shape is obtained.

One way to program such a procedure would be to compile a library of many $p(x)$ vs $r_0(x)$. In addition, to the fact that such a procedure would be costly, there is also the fact that such a procedure is not deterministic. Errors cannot be determined and skill on the part of the designer is still a requirement.

It can be seen that if an algorithm could be found for the comparison box, there exists the possibility that an iterative method, relatively insensitive to the initial guess, could solve the design problem.

In the following sections, the iteration process described previously is put into analytic form and a solution technique is given for the comparison box.

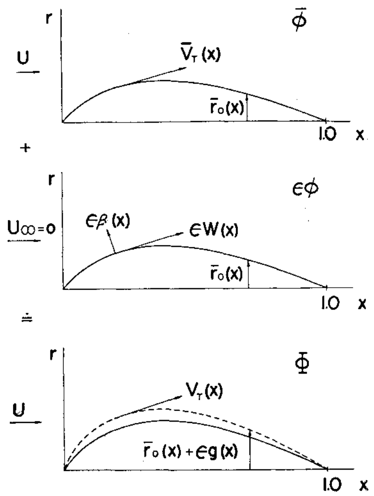


Fig. 3 Blowing distribution.

It is to be noted that the solution for the comparison box, to be described below, immediately provides a solution technique for the development problem which is defined as given an existing body with properties $\bar{r}_0(x)$ and $\bar{p}(x)$. It is desired to change the pressure distribution $\bar{p}(x)$ slightly. What is the corresponding slight change in the shape? This problem occurs in the development phase of the design process since in this phase an existing design is to be changed only slightly.

4. Design Program

The aim is now to program the preceding empirical procedure and thus to solve the free boundary problem, Eqs. (1). To this end, let

$$\begin{aligned} \Phi(x,r;\epsilon) &= \bar{\phi}(x,r) + \epsilon\phi(x,r) \\ V_T(x;\epsilon) &= \bar{V}_T(x) + \epsilon v(x) \\ r_0(x;\epsilon) &= \bar{r}_0(x) + \epsilon g(x) \end{aligned} \qquad (2)$$

where $V_T(x)$ has replaced the role of $p(x)$. In Eqs. (2), a possible ϵ is max $|(V_T - \bar{V}_T)/U| \ll 1$.

Substituting the previous into the original problem, Eqs. (1), and equating like powers of ϵ, as in thin airfoil theory,[8] yields two boundary value problems

$$\begin{aligned} &\text{DE } \Delta\bar{\phi}(x,r) = 0 \\ &\text{BC 1) } r = \bar{r}_0(x){:}\quad \bar{\phi}_{,j}\bar{\nu}_j = 0 \\ &\quad\ \text{2) } x^2 + r^2 \to \infty{:}\quad \bar{\phi} \to Ux \\ &\bar{V}_T = \bar{\phi}_{,j}\bar{\tau}_j \text{ on } r = \bar{r}_0(x) \end{aligned} \qquad (3)$$

$$\begin{aligned} &DE\ \Delta\phi(x,r) = 0 \\ &\text{BC 1) } r = \bar{r}_0(x){:}\quad \phi_{,j}\bar{\nu}_j = \beta(x) \\ &\quad\ \text{2) } x^2 + r^2 \to \infty{:}\quad \phi \to 0 \end{aligned} \qquad (4)$$

$$\phi_{,j}\bar{\tau}_j \equiv W(x) = v(x) - (g(x)/\bar{s}')d(\bar{r}_0'\bar{V}_T/\bar{s}')/dx$$

where $\beta(x) = (1/\bar{s}'\bar{r}_0)\ d(\bar{r}_0\bar{V}_T g(x)/\bar{s}')/dx$, $\bar{r}_0' = dr_0/dx$, $\bar{s}' = d\bar{s}/dx$. Note that $\bar{\nu}_j$ and $\bar{\tau}_j$ are defined on $\bar{r}_0(x)$.

Let $\bar{r}_0(x)$ be the initially guessed body shape, or the shape of an existing body with surface speed $\bar{V}_T(x)$ whose shape is to be changed slightly [i.e., by $\epsilon g(x)$] to produce a slightly different surface speed, $\bar{V}_T + \epsilon v(x)$.

In either case, $\bar{\phi},\bar{r}_0,\bar{V}_T$ are known since Eqs. (3) comprise a solvable problem.

Thus, one is required to solve the problem of Eqs. (4) in which $v(x)$ is given and $g(x)$ is to be found. Note that the free boundary property has been removed. This solution corresponds to the comparison box and is also the solution to the development problem. This particular analysis is presented in the next section.

Using $g(x)$, a corrected body shape $r_0(x) = \bar{r}_0(x) + \epsilon g(x)$ is available and the process is repeated, i.e., Eqs. (3) solved and then Eqs. (4). Analytically,

$$\begin{aligned} r_0^{(Q+1)}(x) &= r_0^{(Q)}(x) + \epsilon g^{(Q)}(x) \\ V_T^{(Q+1)}(x) &= V_T^{(Q)}(x) + \epsilon v^{(Q)}(x) \end{aligned} \qquad (5)$$

where Q is the step of the iteration process with $r_0^{(0)} = \bar{r}_0$, $V_T^{(0)} = \bar{V}_T$, $g^{(0)} = g$ and $v^{(0)} = v$ in the preceding nomenclature.

Equations (2–4) can be illustrated in physical terms as in Fig. 3 where $\beta(x)$ is a blowing distribution.

5. Development Problem

One step in the iteration process proposed to solve the design problem is the solution to Eqs. (4) for $g(x)$, given $v(x)$. This solution is also interpretable as solving the development problem. The solution procedure is given in this section.

If Eqs. (4) are considered with $g(x)$ given and $v(x)$ unknown, the problem is deterministic yielding a unique $v(x)$. One solution technique is that employing surface source distribution of potential theory. Thus, consider a solution for $\phi(x,r)$ in the form

$$\phi(x_j) = -1/4\pi \int_{S(x'_j)} [\sigma(x_j')/l]dS \qquad (6)$$

where $x_j = [x,r]$, $x_j' = [x',r']$, and $l = |x_j - x_j'|$, satisfying the differential equation and the condition at ∞. $\sigma(x_j)$ is the unknown surface source distribution which is determined by the boundary condition on the surface. The equation linking the two is

$$\beta(x) = \phi_{,j}\bar{\nu}_j|_S + \sigma(x)/2 \qquad (7)$$

This is a Fredholm integral equation of the 2nd kind for $\sigma(x)$, given $\beta(x)$. When $\sigma(x)$ has been obtained, the pseudo-surface speed $W(x)$ is obtainable as

$$W(x) = \phi_{,j}\bar{\tau}_j|_S \qquad (8)$$

In Eqs. (7) and (8), $\bar{\nu}_j = [-\sin\alpha, \cos\alpha]$, $\bar{\tau}_j = [\cos\alpha, \sin\alpha]$, where α is the slope of the meridian curve (i.e., $\tan\alpha = d\bar{r}_0/dx$).

The purpose of the Douglas program is to produce numerical solutions to Eqs. (7) and (8) and it does this by reducing the integral equations to algebraic equations.

Thus, let

$$\partial\phi/\partial x = -1/4\pi \int_S \sigma(\partial(1/l)/\partial x)dS \to \sum_{J=1}^{N} X_{IJ}\sigma_J \qquad (9)$$

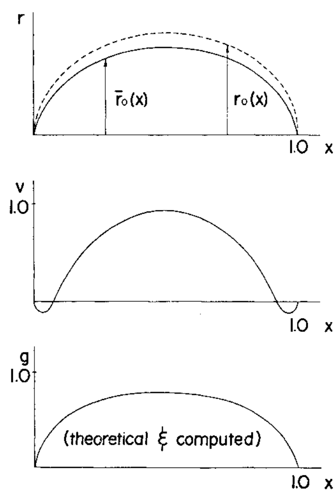


Fig. 4 Test case: spheroid.

$$\partial\phi/\partial r = -1/4\pi \int_S \sigma(\partial(1/l)/\partial r)dS \rightarrow \sum_{J=1}^{N} Y_{IJ}\sigma_J \quad (10)$$

where the body and the flow variables on the surface are approximated by an N-point description. [X_{IJ} and Y_{IJ} are the coefficients of the Douglas program.[3]]

Then Eqs. (7) and (8) reduce to the algebraic equations

$$\beta(x_I) = -\sin\alpha_I \sum_{J=1}^{N} X_{IJ}\sigma_J + \cos\alpha_I \sum_{J=1}^{N} Y_{IJ}\sigma_J + \sigma(X_I)/2 \quad (11)$$

$$W(x_I) = \cos\alpha_I \sum_{J=1}^{N} X_{IJ}\sigma_J + \sin\alpha_I \sum_{J=1}^{N} Y_{IJ}\sigma_J \quad (12)$$

where $I,J = 1,2,\ldots,N$.

Still considering the problem in which $g(x)$ is given, $\beta(x_I)$ is known and Eq. (11) can be solved numerically for $\sigma(x_I)$ which determines $W(x_I)$ in Eq. (12).

Now in the development problem, $g(x)$ is to be found, given $v(x)$. Thus, Eqs. (11) and (12) may be viewed as a set of $2N$ simultaneous algebraic equations for the $2N$ unknowns $\sigma(x_I)$ and $g(x_I)$.

It is to be noted that the development problem is deterministic. In particular if $v = 0$, then $\sigma = g = 0$.

Thus, given a body shape $\bar{r}_0(x)$ with a pressure distribution implied by $\bar{V}_T(x)$, to find the small change in geometry which will induce a slightly different pressure distribution, implied by $\bar{V}_T(x) + \epsilon v(x)$, one solves Eqs. (11) and (12) for $g(x)$ which gives the new body shape $\bar{r}_0(x) + \epsilon g(x)$. This may be checked by solving the direct problem Eq. (1), given $r_0 = \bar{r}_0(x) + \epsilon g(x)$, and compare the resulting $V_T(x)$ with $\bar{V}_T(x) + \epsilon v(x)$. Should more accuracy be required, iterations of the previous scheme are possible.

6. Discussion and Conclusion

A solution technique was presented for the development problem. The solution to the development was presented as one step in an iteration process to solve the design problem. The obvious question to be raised is the convergence of such schemes. Consider first the development problem.

A mathematical proof for the validity of the solution technique presented for the development problem is not available. While the analytical problem is well defined, it is doubtful, owing to its complexity, if one could obtain a proof adequate for practical applications [cf. thin airfoil theory[8]].

It is argued here that the approach should be that which has been employed in thin airfoil theory; namely, in lieu of a mathematical proof, particular checks with exact solutions when available, refinements of the techniques, but basically empirical usage.

In thin airfoil theory checks may be made in cases where the exact solution is known. To provide a check on the solution to the development problem then, an exact solution must be known. The only exact solution in closed form in three dimensional incompressible inviscid flow is that of flow over an ellipsoid of revolution or spheroid.[2]

Thus, consider two ellipsoids of revolution of slightly differing thickness ratios (0.968) and (0.971). For each ellipsoid, the geometry and surface speed satisfy Eqs. (1). Let $\bar{r}_0(x)$ and $\bar{V}_T(x)$ pertain to the first ellipsoid and $\bar{r}_0(x) + \epsilon g(x)$, $\bar{V}_T(x) + \epsilon v(x)$ pertain to the second ellipsoid (where $\epsilon = 0.0018$). Then using $\bar{r}_0(x)$, $\bar{V}_T(x)$, $v(x)$ as input data to compute $g(x)$ using Eqs. (11) and (12), yields a computed $g(x)$ which compares with the exact $g(x)$ within a few percent. [A 77-point subdivision of the x axis was used. For $0.04 < x < 0.96$, the maximum error was less than 2% (including truncation error). For the nose and tail, errors reached 15%. It was seen that this latter figure could have been substantially reduced by a redistribution of the control points.] (see Fig. 4, schematic only).

Since the computed $g(x)$ agreed well with the exact $g(x)$ no iterations were performed [i.e., using $\bar{r}_0(x) + \epsilon g(x)$, $g(x)$ computed, as input to Eqs. (1) would be equivalent to solving the direct problem for $\bar{V}_T + \epsilon v(x)$ for the second ellipsoid].

The computation was carried out at the U.S. Naval Underseas Warfare Center, Pasadena, Calif.

In comparison with thin airfoil theory, then, the previous seems to be the limit of demonstrating the validity of the solution technique for the development problem, a priori. Further checks of a partial nature could be pursued using experimental data but then viscous effects, experimental errors, and the peculiarities of the particular geometry would make the results relevant only to that particular situation and not of a general nature.

Thus, since the solution technique presented seems to be the only one available to date, in its relative degree of rigor, the approach discussed herein would be deemed acceptable in an engineering context.

Consider now the convergence of the iteration process proposed to solve the design problem. The new element is that of the initially guessed $\bar{r}_0(x)$. Thus, the analytical problem of proving convergence of the iteration process in the design solution is not a closed one as opposed to the comparable problem for the development problem. (To define $\bar{r}_0(x)$, a priori, as some function of the given data, if possible, would seem to weaken the technique.)

Thus, as in the case of the development problem, it is argued that the approach to be pursued is that of empirical usage.

References

[1] Baals, D. B., Robins, A. W., and Harris, R. V., "Aerodynamic Design Integration of Supersonic Aircraft," AIAA Paper 68-1018, Philadelphia, Pa., Oct. 1968.

[2] Thwaites, B., ed. "Incompressible Aerodynamics," Clarendon Press, Oxford, 1960, Chap. IX.

[3] Hess, J. L. and Smith, A. M. O., "Calculation of Potential Flow about Arbitrary Bodies," *Progress in Aeronautical Sciences*, Vol. 8, Pergamon, Oxford, 1967.

[4] Young, A. D. and Owen, P. R., "A Simplified Theory for Streamline Bodies of Revolution, and its Application to the Development of High-Speed Low-drag Shapes," R&M 2071, July 1943, British Aeronautical Research Council, London.

[5] Young, A. D. and Owen, E., "A Family of Streamline Bodies of Revolution Suitable for High-Speed or Low-Drag Requirements," R&M 2204, Aug. 1945, British Aeronautical Research Council, London.

[6] McKnown, J. S. and Hsu, E-Y, "Approximation of Axisymmetric Body Form for Specified Pressure Distributions," *Journal of Applied Physics*, Vol. 22, 1951, pp. 864–867.

[7] Munzer, H. and Reichardt, H., "Axially Symmetric Source-Sink Bodies with Predominantly Constant Pressure Distributions," Rept. 6616, UM, 1944 (translated as TM 808–43 1950, U.S. Naval Ordnance Test Station, China Lake, Calif.).

[8] Van Dyke, M., "Perturbation Methods in Fluid Mechanics," Academic Press, New York, 1964, Chap. IV.